Fill Ups, True False of Applications of Derivative

Fill in the Blanks

Q.1. The larger of cos (ln θ) and ln (cos θ) if $e^{-\pi/2} < \theta < \frac{\pi}{2}$ is **.......... (1983 - 1 Mark)**

Ans. Cos (ln θ)

Solution. We have $e^{-\pi/2} < \theta < \pi/2 \Rightarrow -\frac{\pi}{2} < \ln\theta < \ln \pi/2$

⇒ cos (–π/2) < cos (ln θ) < cos (ln π/2)

⇒ cos (ln θ) > 0 .... (1)

Also $-1 \le \cos\theta \le 1 \forall \theta$

$\therefore \quad -1 \le \ln(\cos\theta) \le 0 \forall 0 < \cos\theta \le 1$

⇒ ln (cos q) ≤ 0 .... (2)

From (1) and (2) we get, cos (ln θ) > ln (cos θ)

∴ cos (ln θ) is larger.

Q.2. The function y = 2x^2- ln |x| is monotonically increasing for values of x(≠ 0) satisfying the inequalities .... and monotonically decreasing for values of x satisfying the inequalities .... (1983 - 2 Marks)

Ans. $x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right); \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Solution. $y = 2x^2 - \ln|x| \Rightarrow \frac{dy}{dx} = 4x - \frac{1}{x} = \frac{(2x+1)(2x-1)}{x}$

Critical points are 0, 1/2, –1/2

Clearly f (x) is increasing on $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$ and

If (x) is decreasing on $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$.

Q.3. The set of all x for which ln(1 + x) $\leq$ x is equal to .... (1987 - 2 Marks)

Ans. $x \geq 0$

Solution. Let f (x) = log (1 + x) – x for x > – 1

$$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$$

We observe that,

If ' (x) >0 if – 1 < x < 0 and f '(x) <0 if x > 0

Therefore f increases in (– 1, 0) and decreases in (0, ∞).

Also $f(0) = \log 1 - 0 = 0$

$\therefore \quad x \geq 0 \Rightarrow f(x) \leq f(0)$

$\Rightarrow \quad \log(1+x) - x \leq 0 \Rightarrow \log(1+x) \leq x$

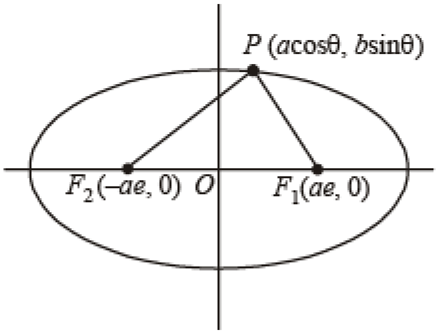
Thus we get, $\log\ (1+x) \leq x, \forall x \geq 0$

Q.4. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2 . If A is the area of the triangle PF_1F_2 then the maximum value of A is.............

(1994 - 2 Marks)

Ans. abe

Solution. Let P(a cos θ,b sin θ) be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 (ae, 0) and F_2 (–ae, 0)



Then area of DPF_1F_2 is given by

$$A=\frac{1}{2}\begin{vmatrix} a\cos\theta & b\sin\theta & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix}$$

$$=\frac{1}{2}|-b\sin\theta(ae+ae)|=abe|\sin\theta|$$

$\because |\sin\theta| \leq 1$

$\therefore A_{max} = abe$

Q.5. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical then H =........... and V = ... (1994 - 2 Marks)

Ans. φ, $\{(1, 1)\}$

Solution. The given curve is C : $y^3 - 3xy + 2 = 0$

Differentiating it with respect to x, we get

$$3y^2\frac{dy}{dx}-3x\frac{dy}{dx}-3y=0 \Rightarrow \frac{dy}{dx}=\frac{y}{-x+y^2}$$

$\therefore$ Slope of tangent to C at point (x_1, y_1) is

$$\frac{dy}{dx}=\frac{y_1}{-x_1+y_1^2}$$

For horizontal tangent, $dy/dx = 0 \Rightarrow y_1 = 0$

For $y_1 = 0$ in C, we get no value of x_1

$\therefore$ There is no point on C at which tangent is horizontal

$\therefore H = \varphi$

For vertical tangent $\frac{dy}{dx}=\frac{1}{0} \Rightarrow -x_1+y_1^2=0 \Rightarrow x_1=y_1^2$

From C, $y_1^3-3y_1^3+2=0$

$\Rightarrow y_1^3=1 \Rightarrow y_1=1 \Rightarrow x_1=1$

$\therefore$ There is only one point (1, 1) at which vertical tangent can be drawn

∴ V = {(1, 1)}

True / False

Q. 1. If x – r is a factor of the polynomial f (x) = $a_n x^4$ + ... + a_0 , repeated m times (1 < m ≤n), then r is a root of f '(x) = 0 repeated m times. (1983 - 1 Mark)

Ans. F

Solution. If (x – r) is a factor of f (x) repeated m times then f ' (x) is a polynomial with (x – r) as factor repeated (m – 1) times.

∴ Statement is false.

Q. 2. For 0 < a < x, th e minimum value of the function $\log_a x + \log_x a$ is 2. (1984 - 1 Mark)

Ans. F

Solution. Given that 0 < a < x.

$$\text{Let } f(x) = \log_a x + \log_x a = \log_a x + \frac{1}{\log_a x} \geq 2$$

But equality holds for $\log_a x = 1$

⇒ x = a which is not possible.

∴ f (x) > 2

∴ f_{min} cannot be 2.

∴ Statement is false.

Subjective Questions of Applications of Derivative (Part - 1)

Q. 1. Prove that the minimum value of $\frac{(a+x)(b+x)}{(c+x)}$, $a, b > c, x > -c$ is $(\sqrt{a-c}+\sqrt{b-c})^2$. **(1979)**

Solution.

$$f(x)=\frac{(a+x)(b+x)}{(c+x)}, a,b>c, x>-c$$
$$=\frac{(a-c+x+c)(b-c+x+c)}{x+c}$$
$$=\frac{(a-c)(b-c)}{x+c}+(x+c)+a+b-2c$$
$$\Rightarrow f'(x)=\frac{-(a-c)(b-c)}{(x+c)^2}+1$$

$$\therefore \quad f'(x)=0 \Rightarrow x=-c\pm\sqrt{(a-c)(b-c)}$$
$$\Rightarrow \quad x=-c+\sqrt{(a-c)(b-c)} \text{ [+ve sign is taken } \because x>-c\text{]}$$
Also $f''(x)=\frac{2(a-c)(b-c)}{(x+c)^3}>0$ for $a, b>c$ and $x>-c$
$$\therefore \quad f(x) \text{ is least at } x=-c+\sqrt{(a-c)(b-c)}$$
$$\therefore \quad f_{min}=\frac{(a-c)(b-c)}{\sqrt{(a-c)(b-c)}}+\sqrt{(a-c)(b-c)}+(a-c)+(b-c)$$

$$=(a-c)+(b-c)+2\sqrt{(a-c)(b-c)}$$
$$=(\sqrt{a-c}+\sqrt{b-c})^2$$

Q. 2. Let x and y be two real variables such that x > 0 and xy = 1. Find the minimum value of x+y. **(1981 - 2 Marks)**

Ans. 2

Solution. Given that x and y are two real variables such that x > 0 and xy = 1.
To find the minimum value of x + y.

Let S = x + y

$$\Rightarrow \quad S = x + \frac{1}{x} \qquad \text{(using } xy = 1)$$

$$\therefore \quad \frac{dS}{dx} = 1 - \frac{1}{x^2}$$

For minimum value of $S, \frac{dS}{dx} = 0$

$$\Rightarrow \quad 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$$

But $x > 0, \therefore x = 1$

Now $\quad \frac{d^2S}{dx^2} = \frac{2}{x^3}$

$$\Rightarrow \quad \left.\frac{d^2S}{dx^2}\right|_{x=1} = 2 = +ve$$

$\therefore \quad S$ is minimum when $x = 1 \therefore S_{\min} = 1 + \frac{1}{1} = 2$

Q. 3. For all x in [0, 1], let the second derivative f ¢¢ (x) of a function f(x) exist and satisfy | f '' (x)| < 1. If f (0) = f (1), then show that | f '(x)| < 1 for all x in [0, 1]. (1981 - 4 Marks)

Solution. We are given that

$$x \in [0,1], |f''(x)| < 1 \text{ and } f(0) = f(1)$$

To prove that $|f'(x)| < 1, \forall x \in [0,1]$

Here f (x) is continuous on [0, 1], differentiable on (0, 1) and f (0) = f (1)

∴ By Rolle's them.,

$\exists c \in (0,1)$ such that $f'(c) = 0$ … (1)

Now there may be three cases for x Î[0,1]

(i) x = c (ii) x > c (iii) x < c

Case I : For x = c.

If x = c then f '(x) = 0<1 [from (1)]

Hence the result | f ' (x) |<1 is obtained in this case.

Case II : For x > c

Consider the interval [c, x].

$\therefore$ By LMV $f''(\alpha)=\dfrac{f'(x)-f'(c)}{x-c}$ where $\alpha\in(c,x)$

$\Rightarrow$ $f'(x)=(x-c)f''(\alpha)$ $[\because f'(c)=0]$

Now, $x,c\in[0,1]$ and $x>c$

$\therefore$ $x-c<1$ .... (i)

also $|f''(x)|<1, \forall x$ (given)

$\therefore$ $|f''(\alpha)|<1$ .... (ii)

Combining (i) and (ii), $(x-c)|f''(\alpha)|<1$

As f '(x) is continuous on [c, x] and differentiable on (c, x)

$\therefore$ | f ' (x) |<1 . Hence the result in this case.

Case III : For x < c Consder the interval [x, c].

As f '(x) is continuous on [x , c] and differentiable on (x, c)

$\therefore$ By LMV for $\beta\in(x,c)$

$$f''(\beta)=\frac{f'(c)-f'(x)}{c-x} \Rightarrow f'(x)=-(c-x)f''(\beta)$$

[Using $f'(c)=0$]

$\therefore$ $|f'(x)|=|(c-x)f''(\beta)|$

as $x,c\in[0,1]$ and $x<c$

$\therefore$ $0<c-x<1$ also $|f''(\beta)|<1$ as $|f''(x)|<1, \not\forall x$

$\therefore$ $|(c-x)f''(\beta)|<1$

$\therefore$ | f ''(x) |<1 hence the result in this case.

Combining all the three cases we get

$|f'(x)|<1, \not\forall x\in[0,1]$

Q. 4. Use the function f (x) =x$^{1/x}$, x > 0. to determine the bigger of the two numbers e$^{\pi}$ and π^{e} **(1981 - 4 Marks)**

Ans. e$^{\pi}$

Solution.

$f(x) = x^{1/x}, \quad x > 0$

Let $y = x^{1/x} \Rightarrow \log y = \frac{1}{x}\log x$

Differentiating w.r.t. x we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{\frac{1}{x}.x - 1.\log x}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y(1-\log x)}{x^2}$$

For max/min value put $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{y(1-\log x)}{x^2} = 0 \Rightarrow \log x = 1 \Rightarrow x = e$$

Also, $$\frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx}(1-\log x) - \frac{1}{x}y\right)x^2 - 2xy(1-\log x)}{x^4}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \left(\frac{-xy}{x^4}\right)_{x=e}$$

$$\left[\text{Using } \frac{\text{dy}}{\text{dx}} = 0, 1-\log x = 0 \; at \; x = e\right]$$

$$= \frac{-e^{1/e}}{e^3} = -ve$$

$\therefore$ y is max at $x = e$

$\therefore$ $e^{1/e}$ is the max. value of $f(x)$.

$\therefore$ $x^{1/x} < e^{1/e}, \forall x$

$\therefore$ Put $x = \pi$, we get, $\pi^{1/\pi} < e^{1/e}$

$\Rightarrow$ Raising to the power peon both sides we get

$\pi^e < e^\pi$ or $e^\pi > \pi^e$

Q. 5. If f(x) and g(x) are differentiable function for $0 \leq x \leq 1$ such that f(0) = 2, g(0) = 0, f(1) = 6; g(1) = 2, then show that there exist c satisfying 0< c < 1 and f ' (c) = 2g' (c). (1982 - 2 Marks)

Solution. Given that f (x) and g (x) are differentiable for x ∈ [0,1] such that f (0) = 2 ; f (1) = 6, g (0) = 0 ; g (1) = 2

To show that $\exists c \in (0,1)$ such that $f'(c) = 2g'(c)$

Let us consider h (x) = f (x) – 2g (x) Then h (x) is continuous on [0, 1] and differentiable on (0, 1)

Also h (0) = f (0) – 2g (0) = 2 – 2 × 0 = 2

h (1) = f (1) – 2g (1) = 6 – 2 × 2 = 2

∴ h (0) = h (1)

∴ All the conditions of Rolle's theorem are satisfied for h(x) on [0, 1]

$\therefore \quad \exists c \in (0,1)$ such that $h'(c) = 0$

$\Rightarrow \; f'(c) - 2g'(c) = 0 \;\; \Rightarrow f'(c) = 2g'(c)$

Q. 6. Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$. (1982 - 2 Marks)

Ans. $\sqrt{c - \frac{1}{4}}$

Solution. (0, c), $y = x^2$, $0 \le c \le 5$.

Any point on parabola is (x, x^2)

Distance between (x, x2) and (0, 1) is

$$D = \sqrt{x^2 + (x^2 - c)^2}$$

To minimum D we consider

$$D^2 = x^4 - (2c-1)x^2 + c^2 = \left(x^2 - \frac{2c-1}{2}\right)^2 + c - \frac{1}{4}$$

Which is minimum when $x^2 - \frac{2c-1}{2} = 0 \Rightarrow x^2 = \frac{2c-1}{2}$

$$\Rightarrow \quad D_{\min} = \sqrt{c - \frac{1}{4}}$$

Q. 7. If $ax^2 + \frac{b}{x} \ge c$ **for all positive x where a > 0 and b > 0 show that**

$27ab^2 \ge 4c^3$. **(1982 - 2 Marks)**

Solution.

Given $ax^2 + \frac{b}{x} \geq c$

$\forall x > 0, a > 0, b > 0$

To show that 27 ab2 $\geq$ 4c$^{3.}$

Let us consider the function f (x) = ax^2 + b/x

then $f'(x) = 2ax - \frac{b}{x^2} = 0$

$\Rightarrow \; x^3 = b/2a \Rightarrow x = (b/2a)^{1/3}$

$\therefore \quad f''(x) = 2a + \frac{2b}{x^3}$

$$\Rightarrow \quad f''\left(\left(\frac{b}{2a}\right)^{1/3}\right) = 2a + \frac{2b}{b} \times 2a = 6a > 0$$

$\therefore \quad f$ is minimum at $x = \left(\frac{b}{2a}\right)^{1/3}$

As (1) is true $\forall \; x$

$\therefore \quad$ so is for $x = \left(\frac{b}{2a}\right)^{1/3}$

$$\Rightarrow \quad a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{(b/2a)^{1/3}} \geq c$$

$$\Rightarrow \quad \frac{a\left(\frac{b}{2a}\right) + b}{(b/2a)^{1/3}} \geq c \quad \Rightarrow \quad \frac{3b}{2}\left(\frac{2a}{b}\right)^{1/3} \geq c$$

As a, b are +ve, cubing both sides we get

$\frac{27b^3}{8} \cdot \frac{2a}{b} \geq c^3 \quad \Rightarrow \quad 27ab^2 \geq 4c^3$ Hence proved.

Q. 8. Show that 1 + x $1 + x \ \ln(x + \sqrt{x^2 + 1}) \geq \sqrt{1 + x^2}$ for all $x \geq 0$ **(1983 - 2 Marks)**

Solution. To show

$$1+x\ln(x+\sqrt{x^2+1})\geq\sqrt{1+x^2}\ \text{ for } x\geq 0$$

$$\text{Consider } f(x)=1+x\ln(x+\sqrt{x^2+1})-\sqrt{1+x^2})$$

$$\text{Here, } f'(x)=\ln(x+\sqrt{x^2+1})+\frac{x}{x+\sqrt{x^2+1}}$$

$$\left[1+\frac{x}{\sqrt{x^2+1}}\right]-\frac{x}{\sqrt{1+x^2}}$$

$$=\ln(x+\sqrt{x^2+1})$$

$$\text{As } x+\sqrt{x^2+1}\geq 1 \text{ for } x\geq 1$$

$$\therefore\quad \ln(x+\sqrt{x^2+1})\geq 0$$

$$\therefore\quad f'(x)\geq 0, \forall x\geq 0$$

Hence f (x) is increasing function.

$$\text{Now for } x\geq 0\Rightarrow f(x)\geq f(0)$$

$$\Rightarrow\quad 1+x\ln(x+\sqrt{x^2+1})-\sqrt{1+x^2}\geq 0$$

$$\Rightarrow\quad 1+x\ln(x+\sqrt{x^2+1})\geq\sqrt{1+x^2}$$

Q. 9. Find the coordinates of the point on the curve $y=\frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope. **(1984 - 4 Marks)**

Ans. (0, 0)

Solution. Equation of the curve is given by

$$y=\frac{x}{1+x^2} \qquad ...(1)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=\frac{1+x^2-x(2x)}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

Again let $f(x)=\frac{1-x^2}{(1+x^2)^2}=\frac{dy}{dx}$

Now, $f'(x)=\frac{(1+x^2)^2(-2x)-(1-x^2)2(1+x^2)(2x)}{(1+x^2)^4}$

$=\frac{(1+x^2)(-2x)-(1-x^2)2.2x}{(1+x^2)^3}=\frac{x(2x^2-6)}{(1+x^2)^3}$

For the greatest value of slope, we have

$f'(x)=\frac{x(2x^2-6)}{(1+x^2)^3}=0 \Rightarrow x=0,\pm\sqrt{3}$

Again we find,

$f''(x)=\frac{12x^2(3-x^2)}{(1+x^2)^4}-\frac{6(1-x^2)}{(1+x^2)^3}$

$\therefore\ f''(0)=-6$ and $f''(\pm\sqrt{3})=\frac{3}{16}$

Thus, second order derivative at x = 0 is negative and second order derivative at x =±√3 is positive.

Therefore, the tangent to the curve has maximum slope at (0, 0).

Q. 10. Find all the tangents to the curve y = cos(x + y), - 2p ≤ x ≤ 2π , that are parallel to the line x + 2y = 0. (1985 - 5 Marks)

Ans. 2x + 4y - π = 0

2x + 4y + 3π = 0

Solution. Equation of given curve y = cos (x + y), -2p ≤ x ≤ 2π

Differentiating with respect to x

$\frac{dy}{dx}=-\sin(x+y).\left[1+\frac{dy}{dx}\right]$

$\Rightarrow [1+\sin(x+y)]\frac{dy}{dx}=-\sin(x+y)$

$\Rightarrow \frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$ ...(1)

Since the tangent to given curve is parallel to x + 2y = 0

$\therefore \quad \frac{-\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$ [For parallel line $m_1 = m_2$]

$\Rightarrow$ 2sin (x + y) = 1 + sin (x + y)

$\Rightarrow$ sin (x + y) = 1

Thus, cos (x + y) = 0

Using equation of curve and above result, we get, y = 0

$\Rightarrow \sin x = 1 \Rightarrow x = n\pi + (-1)^n \pi/2, n \in Z \Rightarrow x = \pi/2, -3\pi/2$
which belong to the interval $[-2\pi, 2\pi]$

Thus the points on curve at which tangents are parallel to given line are (π/2, 0) and (–3π/2, 0)

The equation of tangent at (π/2, 0) is

$$y - 0 = -\frac{1}{2}(x - \pi/2)$$

$\Rightarrow 2y = -x + \pi/2 \Rightarrow 2x + 4y - \pi = 0$
The equation of tangent at (–3π/2, 0) is

$$y - 0 = -\frac{1}{2}(x + 3\pi/2)$$

$\Rightarrow 2y = -x - 3\pi/2 \Rightarrow 2x + 4y + 3\pi = 0$

Thus the required equations of tangents are

$2x + 4y - \pi = 0$ and $2x + 4y + 3\pi = 0$.

Q. 11. Let $f(x) = \sin^3 x + \lambda \sin^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$. **Find the intervals in which λ should lie in order that f(x) has exactly one minimum and exactly one maximum.**
(1985 - 5 Marks)

Ans. $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

Solution. The given function is,

If (x) = $\sin^3$ x + λ $\sin^2$ x for – π/2 < x < π/2

$\therefore$ f ' (x) = 3 $\sin^2$ x cos x + 2λ sin x cos x

$$= \frac{1}{2}\sin 2x(3\sin x + 2\lambda)$$

So, from f ' (x) = 0, we get x = 0 or 3 sin x + 2λ = 0

$$\text{Also, } f''(x) = \cos 2x\,(3\sin x + 2\lambda) + \frac{3}{2}\sin 2x \cos x$$

$$\text{Therefore, for } \lambda = \frac{-3}{2}\sin x \text{, we have}$$

If " (x) = 3 sin x cos2 x = – 2λ cos2 x

Now, if 0 < x < π/2, then – 3/2 < λ < 0 and therefore f " (x) > 0.

⇒ f (x) has one minimum for this value of λ.

Also for x = 0, we have f " (0) = 2λ < 0, That is f (x) has a maximum at x = 0

Again if – p/2 < x < 0, then 0 < λ < 3/2 and therefore f " (x) = – 2λ cos2 x < 0.

So that f (x) has a maximum.

Also for x = 0, f " (a) = 2λ > 0 so that f (x) has a minimum.

Thus, for exactly one maximum and minimum value of f (x),λ must lie in the interval

$$-3/2 < \lambda < 0 \text{ or } 0 < \lambda < 3/2$$

$$\text{i.e., } \lambda \in (-3/2, 0) \cup (0, 3/2).$$

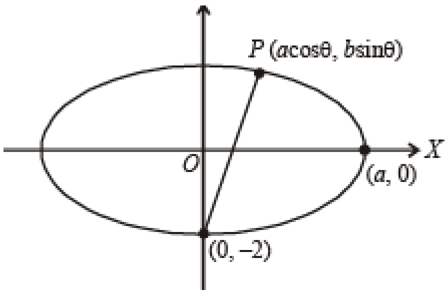
Q. 12. Find the point on the curve $4x^2 + a^2 y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, – 2). (1987 - 4 Marks)

Ans. (0, 2)

Solution. The equation of given curve can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1 \text{ where } 4 < a^2 < 8$$

Clearly it is the question of an ellipse



Let us consider a point P (a cos θ, 2 sin θ) on the ellipse.

Let the distance of P (a cos θ, 2 sin θ) from (0, – 2) is L.

Then, $L^2 = (a \cos \theta - 0)2 + (2 \sin \theta + 2)^2$

⇒ Differentiating with respect to θ, we have

$$\frac{d(L^2)}{d\theta} = \cos\theta[-2a^2 \sin\theta + 8\sin\theta + 8]$$

For max. or min. value of L we should have

$$\frac{d(L^2)}{d\theta} = 0$$

$$\Rightarrow \cos\theta[-2a^2 \sin\theta + 8\sin\theta + 8] = 0$$

$$\Rightarrow \text{Either } \cos\theta = 0$$

$$\text{or } (8 - 2a^2)\sin\theta + 8 = 0 \Rightarrow \theta = \frac{\pi}{2} \text{ or } \sin\theta = \frac{4}{a^2 - 4}$$

$$\text{Since } a^2 < 8 \Rightarrow a^2 - 4 < 4$$

$$\Rightarrow \frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1 \text{ which is not possible}$$

$$\text{Also } \frac{d^2(L^2)}{d\theta^2} = \cos\theta[-2a^2\cos\theta + 8\cos\theta] + (-\sin\theta)[-2a^2\sin\theta + 8\sin\theta + 8]$$

$$\text{At } \theta = \frac{\pi}{2}, \frac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8) < 0$$

as $a^2 < 8$

∴ L is max. at θ = π/2 and the farthest point is (0, 2).

Q. 13. Investigate for maxima and minima the

function $f(x)=\int_1^x [2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]dt$

Ans. f is min at x = 7/5

Solution. We have,

$$f(x)=\int_1^x [2(t-1)(t-2)^3+(t-1)^2 3(t-2)^2]dt$$

Then using the theorem,

$$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} g(t)dt\right]=g[\psi(x)]\psi'(x)-g[\phi(x)]\phi'(x)$$

We get,

$$\begin{aligned} f'(x) &= 2(x-1)(x-2)^3+3(x-1)^2(x-2)^2 \\ &= (x-1)(x-2)^2(2x-4+3x-3) \\ &= (x-1)(x-2)^2(5x-7) \end{aligned}$$

For extreme values f ' (x) = 0 ⇒ x = 1, 2, 7/5

Now, $f''(x) = (x-2)^2(5x-7) + 2(x-1)(x-2)(5x-7) + 5(x-1)(x-2)^2$

At x = 1, f '' (x) = 1 (– 2) = – 2 < 0

∴ f is max. at x = 1

At x = 2 f '' (x) = 0

∴ f is neither maximum nor minimum at x = 2.

At $x = 7/5$

$$f''(x)=5\left(\frac{7}{5}-1\right)\left(\frac{7}{5}-2\right)^2=5\times\frac{2}{5}\times\frac{9}{25}=\frac{18}{25}>0$$

∴ $f(x)$ is minimum at $x = 7/5$.

Q. 14. Find all maxima and minima of the function $y = x(x-1)^2$, $0 \le x \le 2$
(1989 - 5 Marks)

Ans. $\frac{10}{3}$ sq. units

Solution.

We have $y = x(x-1)^2,\ 0 \le x \le 2$

$$\frac{dy}{dx} = (x-1)^2 + 2x(x-1) = (x-1)(3x-1)$$

For max. or min. $\frac{dy}{dx} = 0$

$\Rightarrow\ (x-1)(3x-1) = 0 \Rightarrow x = 1, 1/3$

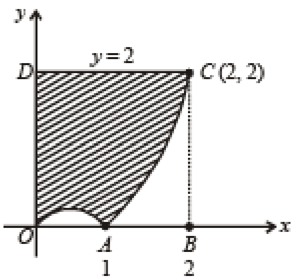
$$\frac{d^2y}{dx^2} = 3x - 1 + 3(x-1) = 6x - 4$$

At $x = 1$, $\frac{d^2y}{dx^2} = 2(+ve)\ \therefore\ y$ is min. at $x = 1$

At $x = 1/3, \frac{d^2y}{dx^2} = -2(-ve)\ \therefore y$ is max. at $x = 1/3$

$\therefore$ Max value of y is $= \frac{1}{3}\left(\frac{1}{3} - 1\right)^2 = \frac{4}{27}$

Min value of y is $= 1(1-1)^2 = 0$



Now the curve cuts the axis x at (0, 0) and (1, 0). When x increases from 1 to 2, y also increases and is +ve.

When y = 2, $x(x-1)^2 = 2$

$\Rightarrow x = 2$

Using max./min. values of y and points of intersection with x-axis, we get the curve as in figure and shaded area is the required area.

$\therefore$ The required area = Area of square $OBCD - \int_0^2 y\,dx$

$$= 2\times 2 - \int_0^2 x(x-1)^2\,dx$$

$$= 4 - \left[\left(x\frac{(x-1)^3}{3}\right)_0^2 - \frac{1}{3}\int_0^2 (x-1)^3 .1\,dx\right]$$

$$= 4 - \left[\frac{x}{3}(x-1)^3 - \frac{(x-1)^4}{12}\right]_0^2$$

$$= 4 - \left[\frac{2}{3} - \frac{1}{12} + \frac{1}{12}\right] = 4 - \frac{2}{3} = \frac{10}{3}\text{sq.units.}$$

Q. 15. Show that 2sin x + tan x $\geq$3x where $0 \leq x < \frac{\pi}{2}$. **(1990 - 4 Marks)**

Solution. Let f (x) = 2 sin x + tan x – 3x on $0 \leq x < \pi/2$

then f ' (x) = 2 cos x + $\sec^2$ x – 3 and f '' (x)

= – 2 sin x + 2 sec2 x tan x

= 2 sin x [$\sec^3$ x – 1]

for $0 \leq x < \pi/2$ $\quad f''(x) \geq 0$

⇒ f ' (x) is an increasing function on $0 \leq x < \pi/2$.

$\therefore$ For $x \geq 0, \Rightarrow f'(x) \geq f'(0)$
$\Rightarrow f'(x) \geq 0$ for $0 \leq x < \pi/2$

⇒ f (x) is an increasing function on $0 \leq x < \pi/2$

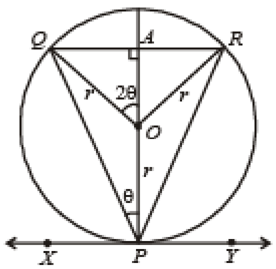
$\therefore$ For $x \geq 0,\ f(x) \geq f(0)$
$\Rightarrow 2\sin x + \tan x - 3x \geq 0,\ 0 \leq x < \pi/2$

$\Rightarrow 2\sin x + \tan x \geq 3x,\ 0 \leq x < \pi/2$ Hence proved

Q. 16. A point P is given on the circumference of a circle of radius r. Chord QR is parallel to the tangent at P. Determine the maximum possible area of the triangle PQR. (1990 - 4 Marks)

Ans. $\frac{3\sqrt{3}}{4}r^2$

Solution. As QR || XY diameter through P is ⊥ QR.



Now area of ΔPQR is given by $A = \frac{1}{2} QR.AP$

But QR = 2.QA = 2r sin 2θ and PA = OA + OP = r cos 2θ + r

$\therefore \quad A = \frac{1}{2}.2r \sin 2\theta.(r + r\cos 2\theta)$

= r^2. 2 sin θ cos θ . 2 $\cos^2$ θ = 4 r^2 sin q $\cos^3$ θ

For max. value of area, $\frac{dA}{d\theta} = 0$

$\Rightarrow \quad 4r^2 [\cos^4\theta - 3\sin^2\theta\cos^2\theta] = 0$

$\Rightarrow \quad \cos^2\theta(\cos^2\theta - 3\sin^2\theta) = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$

Also $\frac{d^2A}{d\theta^2} = 4r^2[-4\cos^3\theta\sin\theta - 6\sin\theta\cos^3\theta$

$+ 6\sin^3\theta\cos\theta]$

$= 4r^2[-10\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$\left.\frac{d^2A}{d\theta^2}\right|_{\theta=30°} = 4r^2\left[-10.\frac{1}{2}.\frac{3\sqrt{3}}{8} + 6.\frac{1}{8}.\frac{\sqrt{3}}{2}\right]$

$= 4r^2\left[\frac{-15\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right] = 4r^2\left(\frac{-12\sqrt{3}}{8}\right) = -ve$

∴ A is maximum at q = 30°

And $A_{max} = 4r^2 \sin 30° \cos^3 30° = 4r^2 \times \frac{1}{2} \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{4} r^2$

Q. 17. A window of perimeter P (including the base of the arch) is in the form of a

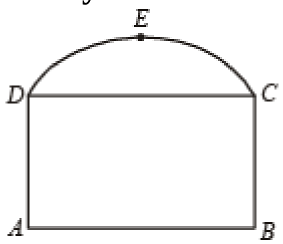
rectangle surrounded by a semi circle. The semi- circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the coloured glass does.
What is the ratio for the sides of the rectangle so that the window transmits the maximum light? (1991 - 4 Marks)

Ans. 6 + π : 6

Solution. Let ABCEDA be the window as shown in the figure and let

AB = x m

BC = y m



Then its perimeter including the base DC of arch

$$= \left(2x + 2y + \frac{\pi x}{2}\right) m$$

$$\therefore \quad P = \left(2 + \frac{\pi}{2}\right)x + 2y \qquad ...(1)$$

Now, area of rectangle ABCD = xy

and area of arch $DCED = \frac{\pi}{2}\left(\frac{x}{2}\right)^2$

Let λ be the light transmitted by coloured glass per sq. m. Then 3λ will be the light transmitted by clear glass per sq. m.

Hence the area of light transmitted $= 3\lambda(xy) + \lambda\left[\frac{\pi}{2}\left(\frac{x}{2}\right)^2\right]$

$$\Rightarrow \quad A = \lambda\left[3xy + \frac{\pi x^2}{8}\right] \qquad ...... (2)$$

Substituting the value of y from (1) in (2), we get

$$A=\lambda\left[3x\frac{1}{2}\left[P-\left(\frac{4+\pi}{2}\right)x\right]+\frac{\pi x^2}{8}\right]$$

$$=\lambda\left[\frac{3Px}{2}-\frac{3(4+\pi)}{4}x^2+\frac{\pi x^2}{8}\right]$$

$$\frac{dA}{dx}=\lambda\left[\frac{3P}{2}-\frac{3(4+\pi)}{2}x+\frac{\pi x}{4}\right]$$

For A to be maximum $\frac{dA}{dx}=01$

$$\Rightarrow \quad x=\frac{\frac{3P}{2}}{\frac{-\pi}{4}+\left(\frac{12+3\pi}{2}\right)}$$

$$\Rightarrow \quad x=\frac{3P}{2}\times\frac{4}{5\pi+24} \Rightarrow \quad x=\frac{6P}{5\pi+24}$$

Also $\frac{d^2A}{dx^2}=\lambda\left[\frac{-3(4+\pi)}{2}+\frac{\pi}{4}\right]<0$

$\therefore$ A is max when $x=\frac{6P}{5\pi+24}$

[Using value of P from (1)]

$$\Rightarrow \quad (5\pi+24-12-3\pi)x=12y \Rightarrow (2\pi+12)x=12y$$

$$\Rightarrow \quad \frac{y}{x}=\frac{\pi+6}{6}$$

∴The required ratio of breadth to length of the rectangle

= 6 + π : 6

Q. 18. A cubic f (x) vanishes at x = 2 and has relative minimum / maximum at $x=-1$ and $x=\frac{1}{3}$ if $\int_{-1}^{1} f\,dx=\frac{14}{3}$, find the **cubic f(x). (1992 - 4 Marks)**

Ans. $x^3 + x^2 - x + 2$

Solution. Let f (x) = $ax^3 + bx^2 + cx + d$

ATQ, f (x) vanishes at x = – 2

⇒ – 8a + 4b – 2c + d = 0 ...(1)

If ' (x) = 3ax2 + 2bx + c

Againg ATQ, f (x) has relative max./min at

$x = -1$ and $x = \frac{1}{3}$

⇒ f ' (– 1) = 0 = f ' (1/3)

⇒ 3a – 2b + c = 0 ... (2)

and a + 2b + 3c = 0 ... (3)

Also, $\int_{-1}^{1} f(x) = \frac{14}{3}$

$$\Rightarrow \left(\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right)_{-1}^{1} = \frac{14}{3}$$

$$\Rightarrow \left[\frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d\right] - \left[\frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d\right] = \frac{14}{3} \Rightarrow \frac{b}{3} + d = \frac{7}{3}$$

$$\Rightarrow b + 3d = 7 \quad \text{...(4)}$$

⇒ b + 3d = 7 ...(4)

From (1), (2), (3), (4) on solving, we get

a = 1, b = 1, c = – 1, d = 2

∴ The required cubic is x^3 + x2 – x + 2.

Q. 19. What normal to the curve $y = x^2$ forms the shortest chord? (1992 - 6 Marks)

Ans. $x + \sqrt{2}y = \sqrt{2}$ or $x - \sqrt{2}y = -\sqrt{2}$

Solution. The given curve is $y = x^2$ ...(1)

Consider any point A (t, t^2) on (1) at which normal chord drawn is shortest.

Then eq. of normal to (1) at A (t, t^2) is

$$y-t^2=-\frac{1}{\left(\frac{dy}{dx}\right)_{(t,t^2)}}(x-t) \quad \left[\text{where } \frac{dy}{dx}=2x \text{ from (1)}\right]$$

$$y-t^2=-\frac{1}{2t}(x-t)$$

$$\Rightarrow \quad x+2ty=t+2t^3 \qquad ...(2)$$

This normal meets the curve again at point B which can be obtained by solving (1) and (2) as follows :

Putting $y = x^2$ in (2), we get

$$2tx^2+x-(t+2t^3)=0,$$
$$D=1+8t(t+2t^3)=1+8t^2+16t^4=(1+4t^2)^2$$

$$\therefore \quad x=\frac{-1+1+4t^2}{4t},\frac{-1-1-4t^2}{4t}=t,-\frac{1}{2t}-t$$

$$\therefore \quad y=t^2,t^2+\frac{1}{4t^2}+1$$

$$\text{Thus, } B\left(-t-\frac{1}{2t},t^2+\frac{1}{4t^2}+1\right)$$

$\therefore$ Length of normal chord

$$AB=\sqrt{\left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2}$$

$$\text{Consider } Z=AB^2=\left(2t+\frac{1}{2t}\right)^2+\left(\frac{1}{4t^2}+1\right)^2$$

$$\Rightarrow \quad Z=\frac{1}{16t^4}+\frac{3}{4t^2}+3+4t^2$$

For shortest chord, we have to minimize Z, and for that dZ/dt =0

$$\Rightarrow\ -\frac{1}{4t^5}-\frac{3}{2t^3}+8t=0\ \Rightarrow\ -1-6t^2+32t^6=0$$

$$\Rightarrow\ 32\,(t^2)^3-6t^2-1=0\ \Rightarrow\ (2t^2-1)(16t^4+8t^2+1)=0$$

$$\Rightarrow\ t^2=\frac{1}{2}\ \text{(leaving –ve values of } t^2)$$

$$\Rightarrow\ t=\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},$$

$$\frac{d^2Z}{dt^2}=\frac{5}{4t^6}+\frac{9}{2t^4}+8$$

$$\left.\frac{d^2Z}{dt^2}\right|_{t=\frac{1}{\sqrt{2}}}=+ve\ \text{also}\ \left|\frac{d^2Z}{dt^2}\right|_{t=-\frac{1}{\sqrt{2}}}=+ve$$

$\therefore$ Z is minimum at $t=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

For $t=\frac{1}{\sqrt{2}}$ normal chord is (from (2)) $x+\sqrt{2}y=\sqrt{2}$

For $t=-\frac{1}{\sqrt{2}}$ normal chord is $x-\sqrt{2}y=-\sqrt{2}$

Q. 20. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at x = 0 **(1993 - 3 Marks)**

Ans. x +y = 1

Solution. The given curve is $y = (1+ x)^y + \sin^{-1}(\sin^2 x)$

Here at x = 0, $y = (1 + 0)y + \sin^{-1}(0) \Rightarrow y = 1$

$\therefore$ Point at which normal has been drawn is (0, 1).

For slope of normal we need to find dy/dx, and for that we consider the curve

as $y=u+v \Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

$$\log u=y\log(1+x)$$

$$\Rightarrow\ \frac{1}{u}\frac{du}{dx}=\frac{y}{1+x}+\log(1+x).\frac{dy}{dx}$$

$$\Rightarrow\ \frac{du}{dx}=(1+x)^y\left[\frac{y}{1+x}+\log(1+x)\frac{dy}{dx}\right]$$

Also $v=\sin^{-1}(\sin^2 x)$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{\sqrt{1-\sin^4 x}}.2\sin x\cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{2\sin x}{\sqrt{1+\sin^2 x}}.$$

Thus, we get,

$$\frac{dy}{dx} = (1+x)^y\left[\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right] + \frac{2\sin x}{\sqrt{1+\sin^2 x}}$$

$$\Rightarrow \quad [1-(1+x)^y \log(1+x)]\frac{dy}{dx}$$

$$= y(1+x)^{y-1} + \frac{2\sin x}{\sqrt{1+\sin^2 x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+x)^{y-1} + \frac{2\sin x}{\sqrt{1+\sin^2 x}}}{1-(1+x)^y \log(1+x)}$$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = 1 \;, \qquad \therefore \quad \text{Slope of normal} = -1$$

∴ Equation of normal to given curve at (0, 1) is

$$y-1 = -1\,(x-0) \Rightarrow \quad x+y=1.$$

Q. 21. Let $f(x) = \begin{cases} -x^3 + \frac{(b^3-b^2+b-1}{(b^2+3b+2)}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at x = 1. (1993 - 5 Marks)

Ans. b ∈ (-2, - 1) ∪ (1,∞)

Solution.

We have, $f(x) = \begin{cases} -x^3 + \frac{b^3-b^2+b-1}{b^3+3b+2}, & 0 \le x < 1 \\ 2x-3, & 1 \le x \le 3 \end{cases}$

We can see from definition of the function, that

f (1) = 2(1) – 3 = – 1

Also f (x) is increasing on [1, 3], f ' (x) being 2 > 0.

∴ f (1) = – 1 is the smallest value of f (x)

Again f ' (x) = – $3x^2$ for x ∈ [0, 1] such that f ' (x) < 0

⇒ f (x) is decreasing on [0, 1]

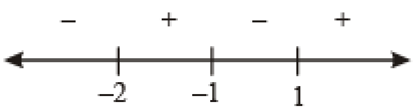
∴ For fixed value of b, its smallest occur when x → 1

$$\text{i.e., } \lim_{h\to 0} f(1-h) = \lim_{h\to 0} -(1-h)^3 + \frac{b^3-b^2+b-1}{b^2+3b+2}$$
$$= -1 + \frac{b^3-b^2+b-1}{b^2+3b+2}$$

As given that the smallest value of f (x) occur at x = 1

∴ Any other smallest value ≥ f (1)

$$\Rightarrow \quad -1+\frac{b^3-b^2+b-1}{b^2+3b+2} \geq -1$$
$$\Rightarrow \quad \frac{b^3-b^2+b-1}{b^2+3b+2} \geq 0 \;\Rightarrow\; \frac{(b^2+1)(b-1)}{(b+2)(b+1)} \geq 0$$
$$\Rightarrow \quad (b-1)(b+1)(b+2) \geq 0$$



⇒ b ∈ (-2, - 1) ∪ (1,∞)

Subjective Questions of Applications of Derivative (Part - 2)

Q.22. The curve y = $ax^3 + bx^2 + cx + 5$, touches the x-axis at P(–2, 0) and cuts the y axis at a point Q, where its gradient is 3. Find a, b, c. (1994 - 5 Marks)

Ans. $a = -\frac{1}{2}, b = \frac{-3}{4}, c = 3$

Solution. Given that y = $ax^3 + bx^2 + cx + 5$ touches the x-axis at P (–2, 0)

$\Rightarrow \left(\frac{dy}{dx}\right)_{x=-2} = 0$ and $P(-2,0)$ lies on curve

$\Rightarrow 3ax^2 + 2bx + c]_{x=-2} = 0$

and – 8a + 4 b – 2c + 5 = 0 ...(2)

[∴ (– 2, 0) lies on curve]

Also the curve cuts the y-axis at Q

∴ For x = 0, y = 5 ∴ Q (0, 5) At Q gradient of the curve is 3

$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 3 \Rightarrow 3x^2 + 2bx + c]_{x=0} = 3$

⇒ c = 3 ...(3)

Solving (1), (2) and (3), we get a = – 1/2, b = – 3/4 and c = 3.

Q.23. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis and the line segment PQ at S.

Find the maximum area of the triangle QSR. (1994 - 5 Marks)

Ans. $\frac{4\sqrt{3}}{9}$ sq. units

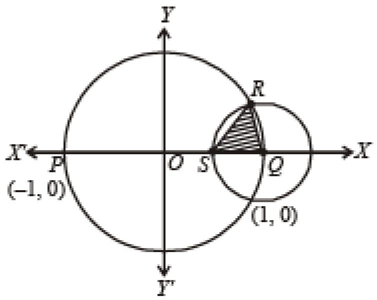
Solution. The given circle is $x^2 + y^2 = 1$ ...(1)

Which intersect x-axis at P (– 1, 0) and Q (1, 0).

Let radius of circle with centre at Q (1, 0) be r, where r is variable.

Then equation of this circle is,

$(x - 1)^2 + y^2 = r^2$ ...(2)



Subtracting (1) from (2) we get

$(x - 1)^2 - x^2 = (r^2 - 1)$

$\Rightarrow \ -2x + 1 = r^2 - 1 \ \Rightarrow \ x = 1 - \frac{r^2}{2}$

Substituting this value of x in (2), we get

$\frac{r^4}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \pm r\sqrt{1 = \frac{r^2}{4}}$

$\therefore \quad R\left(1 - \frac{r^2}{2}, r\sqrt{1 - \frac{r^2}{4}}\right)$ point being above x-axis.

$\therefore$ Area of $\Delta QRS = \frac{1}{2} SQ \times$ ordinate of point R

$\Rightarrow \quad A = \frac{1}{2} \times r \times r\sqrt{1 - \frac{r^2}{4}}$

A will be max. if A^2 is max.

$A^2 = \frac{r^4}{4}\left(1 - \frac{r^2}{4}\right) = \frac{r^4}{4} - \frac{r^6}{16}$

Differentiating A^2 w.r.t. to r, we get $\frac{dA^2}{dr} = r^3 - \frac{3}{8}r^5$

For A^2 to be max. $\frac{dA^2}{dr} = 0$

$$\Rightarrow \quad r^3\left(1-\frac{3}{8}r^2\right) = 0 \Rightarrow r = \frac{2\sqrt{2}}{\sqrt{3}}$$

$$\frac{d^2(A^2)}{dr^2} = 3r^2 - \frac{15}{8}r^4$$

$$\Rightarrow \quad \left.\frac{d^2(A^2)}{dr^2}\right|_{r^2=\frac{8}{3}} = 3\times\frac{8}{3} - \frac{15}{8}\times\frac{64}{9} = -ve$$

$\therefore$ A^2 and hence A is max. when, $r = \frac{2\sqrt{2}}{\sqrt{3}}$

$$\therefore \quad \text{Max. area} = \sqrt{\frac{1}{4}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^4 - \frac{1}{16}\left(\frac{2\sqrt{2}}{\sqrt{3}}\right)^6}$$

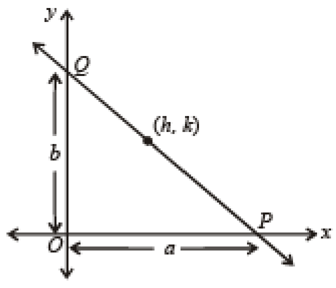
$$= \sqrt{\frac{1}{4}\times\frac{64}{9} - \frac{1}{16}\times\frac{512}{27}} = \sqrt{\frac{16}{9} - \frac{32}{27}}$$

$$= \frac{4}{3\sqrt{3}} = \frac{4\sqrt{3}}{9} \text{ sq. units.}$$

Q.24. Let (h, k) be a fixed point, where h > 0, k > 0. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin. (1995 - 5 Marks)

Ans. 2 kh

Solution. Let the given line be $\frac{x}{a} + \frac{y}{b} = 1$, so that it makes an intercept of a units on x-axis and b units on y-axis. As it passes through the fixed point (h, k), therefore we must have



$$\Rightarrow \quad \frac{k}{b} = 1 - \frac{h}{a} \qquad \Rightarrow \quad b = \frac{ak}{a-h} \qquad \ldots(1)$$

$$\text{Now Area of } \Delta OPQ = A = \frac{1}{2}ab$$

$$\therefore \quad A = \frac{1}{2}a\left(\frac{ak}{a-h}\right) \qquad \text{[using (1)]}$$

$$\text{or} \quad A = \frac{k}{2}\left[\frac{a^2}{a-h}\right]$$

For min. value of A, $\frac{dA}{da} = 0$

$$\Rightarrow \frac{k}{2}\left[\frac{2a(a-h)-a^2}{(a-h)^2}\right] = 0 \Rightarrow \frac{k}{2}\left[\frac{a^2-2ah}{(a-h)^2}\right] = 0 \Rightarrow a = 2h$$

$$\text{Also, } \frac{d^2A}{da^2} = \frac{(2a-2h)(a-h)^2 - 2(a-h)(-1)(a^2-2ah)}{(a-h)^4}$$

$$\therefore \quad \left.\frac{d^2A}{da^2}\right|_{a=2h} = \frac{(2h^3+2h)(0)}{h^4} = \frac{2}{h} > 0, [\because h > 0]$$

$\therefore \quad A$ is min. when $a = 2h$

$$\therefore \quad A_{\min} = \frac{k}{2}\left[\frac{4h^2}{h}\right] = 2kh$$

Q.25. A curve y = f(x) passes through the point P(1,1). The normal to the curve at P is a(y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P. (1996 - 5 Marks)

Ans. $y = e^{a(x-1)}$; 1 sq. unit

Solution. The normal to the curve at P is a (y – 1) + (x – 1) = 0

First we consider the case when a ≠ 0

Slope of normal at P (1 , 1) is $= -\frac{1}{a}$

⇒ Slope of the tangent at (1, 1) is = a

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = a \qquad \ldots (1)$$

But we are given that

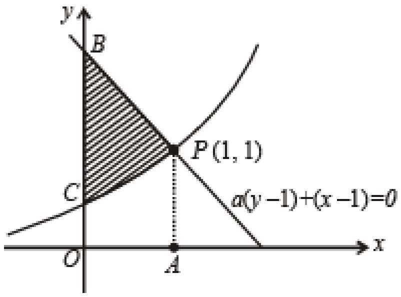
$$\frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = ky \Rightarrow \frac{dy}{y} = kdx$$
$$\Rightarrow \log|y| = kx + C \Rightarrow |y| = e^{kx+c} = e^c . e^{kx}$$
$$\Rightarrow y = \pm e^c e^{kx} \Rightarrow y = Ae^{kx}$$

Where A is constant. As the curve passes through (1, 1)

$$\therefore \quad 1 = Ae^k \quad \Rightarrow \quad A = e^{-k}$$
$$\therefore \quad y = e^{k(x-1)} \quad \Rightarrow \quad \frac{dy}{dx} = ke^{k(x-1)}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = k$$
From (1) and (2), $\left(\frac{dy}{dx}\right)_{1,1} = a = k$

$\therefore$ y = $e^{a(x-1)}$ which is the required curve.

Now the area bounded by the curve, y-axis and normal to curve at (1, 1) is as shown the shaded region in the fig.



$\therefore$ Req. area = ar (PBC) = ar (OAPBCO) – ar (OAPCO)

$$= \int_0^1 y_{\text{normal}} dx - \int_0^1 y_{\text{curve}} dx$$
$$= \int_0^1 \left(-\frac{1}{a}(x-1)+1\right) dx - \int_0^1 e^{a(x-1)} dx$$
$$= \left[-\frac{1}{2a}(x-1)^2 + x\right]_0^1 - \left[\frac{1}{a}e^{a(x-1)}\right]_0^1$$
$$= 1 + \frac{1}{2a} - \frac{1}{a} + \frac{1}{a}e^{-a} = 1 + \frac{1}{a}e^{-a} - \frac{1}{2a}$$

Now we consider the case when a = 0. Then normal at (1, 1) becomes x – 1 = 0 which is parallel to y-axis, therefore tangent at (1, 1) should be parallel to x-axis. Thus

$$\left(\frac{dy}{dx}\right)_{(1,1)} = 0 \qquad \ldots (3)$$

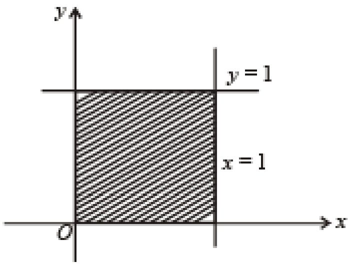
Since $\frac{dy}{dx} \propto y$ gives $y = e^{k(x-1)}$

(as in $a \neq 0$ case)

$\Rightarrow \frac{dy}{dx} = ke^{k(x-1)}$

$\left(\frac{dy}{dx}\right)_{(1,1)} = k$

From (3) and (4), we get k = 0 and required curve becomes y = 1



In this case the required area = shaded area in fig. = 1 sq. unit.

Q.26. Determine the points of maxima and minima of the

function $f(x)=\frac{1}{8}\ \ell n\, x - bx + x^2, x > 0,$ **where b $\geq$ 0 is a constant.** **(1996 - 5 Marks)**

Ans. min at $x = \frac{1}{4}(b + \sqrt{b^2 - 1})$, max at $x = \frac{1}{4}(b - \sqrt{b^2 - 1})$

Solution. $f(x) = \frac{1}{8}\ell nx - bx + x^2, x > 0, b \geq 0$

$$f'(x) = \frac{1}{8x} - b + 2x \qquad \ldots (1)$$

If '(x) = 0 $\Rightarrow$ $16x^2$ - 8bx + 1=0 (for max. or min.)

$$\therefore \quad x = \frac{1}{4}\left[b \pm \sqrt{b^2 - 1}\right] \qquad \ldots (2)$$

Above will give real values of x if $b^2 - 1 \geq 0$ i.e. $b \geq 1$ or $b \leq -1$. But b is given to be +ve. Hence we choose $b \geq 1$

If $b=1$ then $x=\frac{1}{4}$; If $b>1$ then $x=\frac{1}{4}\left[b\pm\sqrt{b^2-1}\right]$

$$f''(x)=-\frac{1}{8x^2}+2=\frac{16x^2-1}{8x^2}$$

Its sign will depend on Nr, $16x^2 - 1$ as $8x^2$ is +ve. We shall consider its sign for $x=\frac{1}{4}$ and $x=\frac{1}{4}\left[b\pm\sqrt{b^2-1}\right]$

$$N^r \text{ of } f''(x)=16x^2-1=[b+\sqrt{b^2-1}]^2-1$$
$$=+\text{ve for } b>1 \quad \therefore \text{Minima}$$

or $\quad N^r$ of $f''(x)=(b-\sqrt{b^2-1})^2-1$
$$=-\text{ve for } b>1 \quad \therefore \text{Maxima}$$

Q.27. Let $f(x)=\begin{cases} xe^{ax}, & x\le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$

Where a is a positive constant. Find the interval in which f ' (x) is increasing. (1996 - 3 Marks)

Ans. $\left(\frac{-2}{a},\frac{a}{3}\right)$

Solution. Given that, $f(x)=\begin{cases} xe^{ax}, & x\le 0 \\ x+ax^2-x^3, & x>0 \end{cases}$

Differentiating both sides, we have

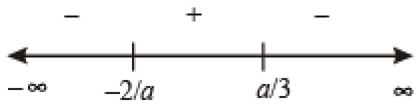
$$f'(x)=\begin{cases} axe^{ax}+e^{ax}, & x\le 0 \\ 1+2ax-3x^2, & x>0 \end{cases}$$

Again differentiating both sides, we have

$$f''(x)=\begin{cases} 2ae^{ax}+a^2\,x\,e^{ax}; & x\le 0 \\ 2a-6x; & x>0 \end{cases}$$

For critical points, we put f "(x) =0

$$\Rightarrow \quad x = -\frac{2}{a}, \text{if } x \le 0 \;= \frac{a}{3}, \text{if} x > 0$$



It is clear from number line that

$$f''(x) \text{ is +ve on } \left(-\frac{2}{a}, \frac{a}{3}\right)$$
$$\Rightarrow \quad f'(x) \text{ increases on } \left(-\frac{2}{a}, \frac{a}{3}\right)$$

Q.28. Let a + b = 4, where a < 2, and let g(x) be a differentiable function. (1997 - 5 Marks)

If $\frac{dg}{dx} > 0$ **for all x, prove that** $\int_0^a g(x)\,dx + \int_0^b g(x)dx$

Solution.

Let $b - a = t$, where $a + b = 4$

$$\Rightarrow \quad a = \frac{4-t}{2} \text{ and } b = \frac{t+4}{2}$$

as given $a < 2$ and $b > 2 \Rightarrow t > 0$

Now $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$

$$= \int_0^{\frac{4-t}{2}} g(x)\,dx + \int_0^{\frac{4+t}{2}} g(x)\,dx = \phi(t) \text{ [say]}$$

$$\Rightarrow \phi'(t) = g\left(\frac{4-t}{2}\right)\left(-\frac{1}{2}\right) + g\left(\frac{4+t}{2}\right)\left(\frac{1}{2}\right) \text{ NOTE THIS STEP}$$

$$\left[\text{Using} \frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)\,dt\right] = f[v(x)].v'(x) - f[u(x)]u'(x)\right]$$

$$= \frac{1}{2}\left[g\left(\frac{4+t}{2}\right) - g\left(\frac{4-t}{2}\right)\right]$$

Since g (x) is an increasing function (given)

$\therefore \quad$ for $x_1 > x_2 \Rightarrow g(x_1) > g(x_2)$

Here we have $\left(\frac{4+t}{2}\right) > \left(\frac{4-t}{2}\right)$

$\Rightarrow \quad g\left(\frac{4+t}{2}\right) > g\left(\frac{4-t}{2}\right)$

$\Rightarrow \quad \phi'(t) = \frac{1}{2}\left[g\frac{(4+t)}{2} - g\frac{(4-t)}{2}\right] > 0 \Rightarrow \phi'(t) > 0$

Hence φ (t) increase as t increases.

$\Rightarrow \quad \int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b-a)$ increases.

Q.29. Suppose f(x) is a function satisfying the following conditions (1998 - 8 Marks)

(a) f(0) = 2, f(1) = 1,

(b) f has a minimum value at x = 5/2, and
(c) for all x,

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Where a, b are some constants. Determine the constants a, b and the function f(x).

Ans. $a = \frac{1}{4}, b = \frac{-5}{4}, c = 2 \quad , f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

Solution. Applying $R_3 \to R_3 - R_1 - 2R_2$ we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-a & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [\text{Using } C_2 \to C_2 - C_1]$$

$\Rightarrow \quad f'(x) = 2ax + b$

$\Rightarrow$ f '(x) = 2ax+b

Integrating, we get , f (x) = ax2 + bx + C where C is an arbitrary constant. Since f has a maximum at x = 5/2,

If '(5 / 2) = 0 ⇒ 5a +b=0 … (1)

Also f (0) = 2 ⇒ C = 2

And f (1) = 1 ⇒ a + b + c = 1

∴ a + b = – 1 … (2)
Solving (1) and (2) for a, b we get,

a = 1/4, b = – 5/4

$a = 1/4, b = -5/4$

Thus, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2.$

Q. 30. A curve C has the property that if the tangent drawn at any point P on C meets the co-ordinate axes at A and B, then P is the mid-point of AB. The curve passes through the point (1, 1). Determine the equation of the curve.
(1998 - 8 Marks)

Ans. xy = 1

Solution. Equation of the tangent at point (x, y) on the curve is

$$Y - y = \frac{dy}{dx}(X - x)$$

This meets axes in

$$A\left(x - y\frac{dx}{dy}, 0\right) \text{ and } B\left(0, y - x\frac{dy}{dx}\right)$$

Mid-point of AB is $\left(\frac{1}{2}\left(x - y\frac{dx}{dy}\right), \frac{1}{2}\left(y - x\frac{dy}{dx}\right)\right)$

We are given

$$\frac{1}{2}\left(x-y\frac{dx}{dy}\right)=x \text{ and } \frac{1}{2}\left(y-x\frac{dy}{dx}\right)=ya$$

$$\Rightarrow \quad x\frac{dy}{dx}=-y \Rightarrow \frac{dy}{y}=-\frac{dx}{x}$$

Intergrating both sides,

$$\int\frac{dy}{y}=-\int\frac{dy}{x}\Rightarrow \log y=-\log x+c$$

Put x = 1, y = 1,

⇒ log 1 = – log 1 + c ⇒ c = 0 ⇒ log y + log x = 0

⇒ log yx = 0 ⇒ yx = e^0 = 1

Which is a rectangular hyperbola.

Q. 31. Suppose p(x) = $a^0 + a_1x + a_2x^2 + ... + a_nx^n$. If $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$, prove that $|a_1+2a_2+.....+na_n| \le 1$. **(2000 - 5 Marks)**

Solution. Given that, $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_nx^n$ … (1)

and $|p(x)| \le |e^{x-1}-1|, \ \forall\ x \ge 0$

To prove that,

$$|a_1+2a_2+\ldots+n\,a_n| \le 1$$

It can be clearly seen that in order to prove the result it is sufficient to prove that | p '(1) | ≤ 1

We know that,

$$|p'(1)| = \lim_{h\to 0}\left|\frac{p(1+h)-p(1)}{h}\right|$$
$$\le \lim_{h\to 0}\frac{|p(1+h)|+|p(1)|}{|h|}$$

[Using $|x-y| \le |x|+|y|$]

But $|p(1)| \le |e^0-1|$ [Using equation (2) for x = 1]

⇒ | p (1) | ≤ 0

But being absolute value, | p(1) | $\geq$ 0 .

Thus we must have | p(1) | =0

$$\text{Also } |p(1+h)| \leq |e^h - 1| \qquad (\text{Using eq}^n\ (2) \text{ for } x = 1+h)$$

$$\text{Thus } |p'(1)| \leq \lim_{h\to 0} \frac{|e^h - 1|}{|h|} = 1$$

$$\text{or} \quad |p'(1)| \leq 1 \Rightarrow |a_1 + 2a_2 + \ldots\ldots + na^n| \leq 1$$

Q. 32. Let - 1 $\leq$ p $\leq$ 1. Show that the equation $4x^3$ - 3x - p = 0 has a unique root in the interval [1/2, 1] and identify it. (2001 - 5 Marks)

Solution. Given that -1 $\leq$ p $\leq$ 1.

Consider f (x) = $4x^3 - 3x - p = 0$

$$\text{Now, } f(1/2) = \frac{1}{2} - \frac{3}{2} - p = -1 - p \leq 0 \text{ as } (-1 \leq p)$$

$$\text{Also } f(1) = 4 - 3 - p = 1 - p \geq 0 \text{ as } (p \leq 1)$$

$\therefore$ f (x) has at least one real root between [1/2, 1].

Also f '(x) = $12x^2$ - 3>0 on [1/2, 1]

$\Rightarrow$ f is increasing on [1/2, 1]

$\Rightarrow$ f has only one real root between [1/2, 1]

To find the root, we observe f (x) contains $4x^3 - 3x$ which is multiple angle formula of cos 3θ if we put x = cos θ.

$\therefore$ Let the req. root be cos θ then,

$$4\cos^3\theta - 3\cos\theta - p = 0$$

$$\Rightarrow \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1} p \Rightarrow \theta = \frac{1}{3}\cos^{-1}(p)$$

$$\therefore \text{ Root is } \cos\left(\frac{1}{3}\cos^{-1}(p)\right).$$

Q. 33. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line x + y = 7, is minimum. (2003 - 2 Marks)

Ans. (2, 1)

Solution. The given curve is $\frac{x^2}{6}+\frac{y^2}{3}=1$ (an ellipse)

Any parametric point on it is $P(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$.

Its distance from line x + y = 7 is given by

$$D=\frac{\sqrt{6}\cos\theta+\sqrt{3}\sin\theta-7}{\sqrt{2}}$$

For min. value of D, $\frac{dD}{d\theta}=0$

$\Rightarrow \quad -\sqrt{6}\sin\theta+\sqrt{3}\cos\theta=0 \quad \Rightarrow \quad \tan\theta=1/\sqrt{2}$

$\Rightarrow \quad \cos\theta=\frac{\sqrt{2}}{\sqrt{3}}$ and $\sin\theta=\frac{1}{\sqrt{3}}$

$\therefore$ Required point P is (2, 1)

Q. 34. Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that sin (tan x) $\geq x, \ \forall\ x \in \left[0, \frac{\pi}{4}\right]$ **(2003 - 4 Marks)**

Solution. Given that $2(1 - \cos x) < x^2$, $x \neq 0$

To prove $\sin(\tan x) \geq x$, $x \in [0, \pi/4)$.

Let us consider $f(x) = \sin(\tan x) - x$

$\Rightarrow f'(x) = \cos(\tan x)\sec^2 x - 1$

$$=\frac{\cos(\tan x)-\cos^2 x}{\cos^2 x}$$

As given $2(1-\cos x)<x^2,\ x\neq 0$

$\Rightarrow \quad \cos x>1-\frac{x^2}{2}$

Similary, $\cos(\tan x) > 1 - \frac{\tan^2 x}{2}$

$$\therefore \quad f'(x) > \frac{1 - \frac{1}{2}\tan^2 x - \cos^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x\left[1 - \frac{1}{2\cos^2 x}\right]}{\cos^2 x}$$

$$= \frac{\sin^2 x(\cos 2x)}{2\cos^4 x} > 0, \forall x \in [0, \pi/4)$$

∴ f '(x) > 0 ⇒ f (x) is an increasing function.

$\therefore$ For $x \in [0, \pi/4)$,

$x \geq 0 \Rightarrow f(x) \geq f(0)$

$\Rightarrow \sin(\tan x) - x \geq \sin(\tan 0) - 0$

$\Rightarrow \sin(\tan x) - x \geq 0$

$\Rightarrow \sin(\tan x) \geq x$ Hence proved.

Q. 35. If the function f : [0,4] → R is differentiable then show that (2003 - 4 Marks)

(i) For a, b ∈ (0,4), $(f(4))^2 - (f(0))^2 = 8f'(a)\, f(b)$

(ii) $\int_0^4 f(t)dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)] \forall\ 0 < \alpha, \beta < 2$

Solution. Given that f is a differentiable function on [0, 4]

∴ It will be continuous on [0, 4]

∴ By Lagrange's mean value theorem, we get

$$\frac{f(4) - f(0)}{4 - 0} = f'(a), \text{ for } a \in (0, 4) \quad \ldots(1)$$

Again since f is continuous on [0, 4] by intermediate mean value theorem, we get

$$\frac{f(4) + f(0)}{2} = f(b) \text{ for } b \in (0, 4) \quad \ldots(2)$$

[If f (x) is continuous on $[\alpha, \beta]$ then $\exists \mu \in (\alpha, \beta)$

such that $f(\mu) = \dfrac{f(\alpha) + f(\beta)}{2}$]

Multiplying (1) and (2) we get

$$\frac{[f(4)]^2 - [f(0)]^2}{8} = f'(a) f(b);\ a, b \in (0, 4)$$

or $[f(4)]^2 - [f(0)]^2 = 8 f'(a) f(b)$

Hence Proved.

(ii) To prove

$$\int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]\ \forall\ 0 < \alpha, \beta < 2$$

Let $I = \int_0^4 f(t)dt$

Let $t = u^2$ also $t \to 0 \Rightarrow u \to 0$

$\Rightarrow\ dt = 2u\,du$ as $t \to 4 \Rightarrow u \to 2$

$$\therefore\ \int_0^4 f(t)\,dt = \int_0^2 f(u^2).2u\,du \qquad \ldots(1)$$

Consider, $F(x) = \int_0^x f(u^2).2u\,du$

Then clearly F (x) is differentiable and hence continuous on [0, 2]

By LMV theorem, we get some, $\mu \in (0, 2)$

such that $F'(\mu) = \dfrac{F(2) - F(0)}{2 - 0}$

$$\Rightarrow\ f(\mu^2).2\mu = \frac{\int_0^2 f(u^2)\,2u\,du}{2} \ldots(2)$$

Again by intermediate mean value theorem,

$\exists\ \alpha, \beta$ such that $0 < \alpha < \mu < \beta < 2$

$\Rightarrow\quad F'(\mu) = \dfrac{F'(\alpha) + F'(\beta)}{2}$, as f is continuous on [0, 2]

$\Rightarrow\quad F$ is continuous on [0, 2]

$$\Rightarrow\quad f(\mu^2).2\mu = \frac{f(\alpha^2)2\alpha + f(\beta^2).2\beta}{2}$$

$$\Rightarrow\quad f(\mu^2).2\mu = \alpha f(\alpha^2) + \beta f(\beta^2) \qquad \ldots(3)$$

From (2) and (3), we get

$$\int_0^2 f(u^2)\,2u\,du = 2[\alpha\, f(\alpha^2) + \beta\, f(\beta^2)]$$

where $0 < \alpha, \beta < 2$

$$\int_0^4 f(t)\,dt = 2[\alpha\, f(\alpha^2) + \beta\, f(\beta^2)]$$

where $0 < \alpha, \beta < 2$ (Using eq^n (1))

Hence Proved.

Q. 36. If $P(1) = 0$ and $\dfrac{dP(x)}{dx} > P(x)$ **for all x $\geq$ 1 then prove that P(x) > 0 for all x > 1.** **(2003 - 4 Marks)**

Solution. We are given that,

$$\frac{dP(x)}{dx} > P(x), \forall, x \geq 1 \text{ and } P(1) = 0$$

$$\Rightarrow\quad \frac{dP(x)}{dx} - P(x) > 0$$

Multiplying by e^{-x}, we get,

$$e^{-x}\frac{dP(x)}{dx} - e^{-x}P(x) > 0$$

$$\Rightarrow\quad \frac{d}{dx}\left[e^{-x}P(x)\right] > 0$$

$\Rightarrow$ e^{-x} P(x) is an increasing function.

$\therefore\quad \forall\, x > 1, e^{-x}P(x) > e^{-1}P(1) = 0 \qquad$ [Using $P(1) = 0$]

$\Rightarrow\quad e^{-x}P(x) > 0, \forall\, x > 1$

$\Rightarrow\quad P(x) > 0, \forall\, x > 1 \qquad [\because e^{-x} > 0]$

Q. 37. Using Rolle 's Theorem, prove that there is at least one root in ($45^{1/100}$, 46) of the polynomial **(2004 - 2 Marks)**

P(x) = $51x^{101}$ – $2323(x)^{100}$ – 45x + 1035.

Solution. We are given,

P(x) = $51x^{101} - 2323(x)^{100} - 45x + 1035$.

To show that at least one root of P (x) lies in ($45^{1/100}$, 46), using Rolle's theorem, we consider antiderivative of P (x)

$$\text{i.e. } F(x) = \frac{x^{102}}{2} - \frac{2323x^{101}}{101} - \frac{45x^2}{2} + 1035x$$

Then being a polynominal function F(x) is continuous and differentiable.

$$\text{Now, } F(45^{1/100}) = \frac{(45)^{\frac{102}{100}}}{2} - \frac{2323(45)^{\frac{101}{100}}}{101} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}}$$

$$= \frac{45}{2}(45)^{\frac{2}{100}} - 23 \times 45(45)^{\frac{1}{100}} - \frac{45.(45)^{\frac{2}{100}}}{2} + 1035(45)^{\frac{1}{100}} = 0$$

$$\text{And } F(46) = \frac{(46)^{102}}{2} - \frac{2323(46)^{101}}{101} - \frac{45(46)^2}{2} + 1035(46)$$

$$= 23\,(46)^{101} - 23\,(46)^{101} - 23 \times 45 \times 46 + 1035 \times 46 = 0$$

$$\therefore \quad F(45^{\frac{1}{100}}) = F(46) = 0$$

∴ Rolle's Theorem is applicable.

Hence, there must exist at least one root of F '(x) = 0

i.e. P (x) = 0 in the interval $\left(45^{\frac{1}{100}}, 46\right)$

Q. 38. Prove that for $x \in \left[0, \frac{\pi}{2}\right],\ \sin x + 2x \geq \frac{3x(x+1)}{\pi}$. **Explain the identity if any used in the proof.** **(2004 - 4 Marks)**

Solution. Let us consider,

$$f(x) = \sin x + 2x - \frac{3x(x+1)}{\pi}$$

$$\Rightarrow \quad f'(x) = \cos x + 2 - \frac{3}{\pi}(2x+1)$$

$$\Rightarrow \quad f''(x) = -\sin x - \frac{6}{\pi} < 0, \forall\, x \in [0, \pi/2]$$

⇒ f '(x) is a decreasing function. .... (1)

Also $f'(0) = 3 - \frac{3}{\pi} > 0$ … (2)

and $f'(\pi/2) = 2 - \frac{3}{\pi}(\pi+1) = -1 - \frac{3}{\pi} < 0$ …(3)

Equations (1), (2) and (3) shows that.

⇒ There exists a certain value of x ∈ [0,π / 2] for which f ' (x) = 0 and this point must

be a point of maximum for f (x) since the sign of f ' (x) changes from +ve to –ve.

Also we can see that f (0) = 0 and

$$f\left(\frac{\pi}{2}\right) = \pi + 1 - \frac{3}{2}\left(\frac{\pi}{2} + 1\right) = \frac{\pi}{4} - \frac{1}{2} > 0$$

Let x = p be the point at which the max. of f (x) occurs.

There will be only one max. Point in [0, π/2]. Since f '(x) = 0 is only once in the interval.

Consider, x ∈ [0,p]

⇒ f ' (x) > 0 ⇒ f (x) is an increasing function.

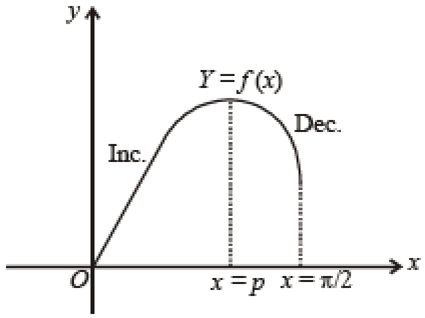
$$\Rightarrow \quad f(0) \le f(x) \; [as\ 0 \le x]$$

$$\Rightarrow \quad f(x) \ge 0$$ ..... (4)

Also for $x \in [p, \pi/2]$

$\Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing function.

$\Rightarrow$ for $x < \pi/2, f(x) > f(\pi/2) > 0$ ..... (5)



Hence from (4) and (5) we conclude that

$f(x) \geq 0, \forall x \in [0, \pi/2]$.

Q. 39. If $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the cuve y = f (x) at the point (1, 2). (2005 - 2 Marks)

Ans. y = 2

Solution. Given that, $|f(x_1) - f(x_2)| < (x_1 - x_2)^2$, $x_1, x_2 \in R$

Let $x_1 = x + h$ and $x_2 = x$ then we get

$$|f(x+h) - f(x)| < h^2 \Rightarrow |f(x+h) - f(x)| < |h|^2$$

$$\Rightarrow \left|\frac{f(x+h) - f(x)}{h}\right| < |h|$$

Taking limit as $h \to 0$ on both sides, we get

$$\lim_{h \to 0}\left|\frac{f(x+h) - f(x)}{h}\right| < \delta \text{ (a small +ve number)}$$

$$\Rightarrow |f'(x)| < \delta \Rightarrow f'(x) = 0$$

$\Rightarrow$ f (x) is a constant function. Let f (x) = k i.e., y = k

As f (x) passes through (1, 2) $\Rightarrow$ y = 2

∴ Equation of tangent at (1, 2) is,

$y - 2 = 0 (x - 1)$ i.e. $y = 2$

Q. 40. If p(x) be a polynomial of degree 3 satisfying p(–1) = 10, p(1) = –6 and p(x) has maxima at x = – 1 and p'(x) has minima at x = 1. Find the distance between the local maxima and local minima of the curve. (2005 - 4 Marks)

Ans. $4\sqrt{65}$

Solution. Let $p(x) = ax^3 + bx^2 + cx + d$

$p(-1) = 10$

$\Rightarrow -a + b - c + d = 10$ ...... (i)

$p(1) = -6$

$\Rightarrow a + b + c + d = -6$...... (ii)

p (x) has max. at $x = -1$

$\therefore p'(-1) = 0$

$\Rightarrow 3a - 2b + c = 0$...... (iii)

p' (x) has min. at $x = 1$

$\therefore p''(1) = 0$

$\Rightarrow 6a + 2b = 0$...... (iv)

Solving (i), (ii), (iii) and (iv), we get

From (iv), $b = -3a$

From (iii), $3a + 6a + c = 0 \Rightarrow c = -9a$

From (ii), $a - 3a - 9a + d = -6 \Rightarrow d = 11a - 6$

From (i), $-a - 3a + 9a + 11a - 6 = 10$

$\Rightarrow$ 16a = 16 $\Rightarrow$ a = 1 $\Rightarrow$ b = – 3, c = – 9, d = 5

$\therefore$ p (x) = $x^3 - 3x^2 - 9x + 5$

$\Rightarrow$ p' (x) = $3x^2 - 6x - 9 = 0$

$\Rightarrow$ 3 (x + 1) (x – 3) = 0 $\Rightarrow$ x = – 1 is a point of max. (given) and x = 3 is a point of min.

[$\therefore$ Max. and min. occur alternatively]

$\therefore$ points of local max. Is (– 1, 10) and local min. is (3, – 22).

And distance between them is

$$= \sqrt{[3-(-1)^2] + (-22-10)^2}$$

$$= \sqrt{16+1024} = \sqrt{1040} = 4\sqrt{65}$$

Q. 41. For a twice differentiable function f (x), g(x) is defined as g(x) = (f '(x)^2 + f"(x)) f(x) on [a, e]. If for a < b < c < d < e, f (a) = 0, f (b) = 2, f (c) = –1, f (d) = 2, f (e) = 0 then find the minimum number of zeros of g(x). (2006 - 6M)
Ans. 6

Solution. $g(x) = (f'(x))^2 + f''(x)f(x) = \frac{d}{dx}(f(x)f'(x))$

Let h (x) = f (x) f ' (x)

Then, f (x) = 0 has four roots namely a, a, b, e

Where b < a < c and c < b < d.
And f ' (x) = 0 at three points k_1, k_2, k_3

where a < k_1 < α, α < k_2 < β, β < k_3 < e

[$\therefore$ Between any two roots of a polynomial function f (x) = 0 there lies at least one root of f ' (x) = 0]

$\therefore$ There are at least 7 roots of f (x). f ' (x) = 0

$\Rightarrow$ There are at least 6 roots of $\frac{d}{dx}(f(x)f'(x)) = 0$
i.e. of g (x) = 0

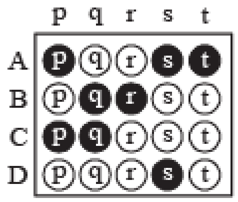
Additional Questions of Applications of Derivatives

Match the Following

This question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t.

Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following

example :
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. In this questions there are entries in columns I and II. Each entry in column I is related to exactly one entry in column II. Write the correct letter from column II against the entry number in column I in your answer book.

Let the functions defined in column I have domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Column I	**Column II**
(A) x + sin x	**(p) increasing**
(B) sec x	**(q) decreasing**
	(r) neither increasing nor decreasing

Ans. (A) → p (B) → r

Solution. (A) f (x) = x + sin x on (–π/2, π/2)

f ' (x) = 1 + cos x

As 0 ≤ cos x ≤ 1 for x ∈ (–π/2, π/2)

∴ f ' (x) > 0 on (–π/2, π/2)

(A) $\rightarrow$ p

(B) f (x) = sec x $\Rightarrow$ f ' (x) = sec x tan x.

Clearly f ' (x) < 0 in $(-\pi/2, 0)$ and f ' (x) > 0 in $(0, \pi/2)$

$\therefore$ On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) f(x)$ is neither increasing non decreasing.

(B) $\rightarrow$ r

Integer Value Correct Type

Q. 1. The maximum value of the function
f (x) = $2x^3 - 15x^2 + 36x - 48$ on the set
A= $\{x \mid x^2 + 20 \le 9x\}$ is

Ans. 7

Solution. The given function is f (x) = $2x^3 - 15x^2 + 36x - 48$

and A = $\{x \mid x^2 + 20 \le 9x\}$

$\Rightarrow A = \{x \mid x^2 - 9x + 20 \le 0\}$

$\Rightarrow A = \{x \mid (x - 4)(x - 5) \le 0\} \Rightarrow A = [4, 5]$

Also $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

Clearly $\forall\ x \in A,\ f'(x) > 0$

$\therefore$ f is strictly increasing function on A.

$\therefore$ Maximum value of f on A

$= f(5) = 2 \times 5^3 - 15 \times 5^2 + 36 \times 5 - 48$

$= 250 - 375 + 180 - 48 = 430 - 423 = 7$

Q. 2. Let p(x) be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2.$

Then the value of p (2) is

Ans. 0

Solution. Let $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

Now $\lim_{x\to 0}\left[1+\frac{p(x)}{x^2}\right] = 2$

$\Rightarrow \lim_{x\to 0}\frac{p(x)}{x^2} = 1$ ...(1)

$\Rightarrow p(0) = 0 \Rightarrow e = 0$

Applying L'Hospital's rule to eq^n (1), we get

$\lim_{x\to 0}\frac{p'(x)}{2x} = 1 \Rightarrow p'(0) = 0$

$\Rightarrow d = 0$

Again applying L 'Hospital's rule, we get

$\lim_{x\to 0}\frac{p''(x)}{2} = 1 \Rightarrow p''(0) = 2$

$\Rightarrow 2c = 2$ or $c = 1$

$\therefore p(x) = ax^4 + bx^3 + x^2$

$\Rightarrow p'(x) = 4ax^3 + 3bx^2 + 2x$

As p(x) has extremum at x = 1 and 2

$\therefore p'(1) = 0$ and $p'(2) = 0$

$\Rightarrow 4a + 3b + 2 = 0$ ...(i)

$\Rightarrow 32a + 12b + 4 = 0$ or $8a + 3b + 1 = 0$ ...(ii)

Solving eq's (i) and (ii) we get $a = \frac{1}{4}$ and $b = -1$

$\therefore p(x) = \frac{1}{4}x^4 - x^3 + x^2$

So, that $p(2) = \frac{16}{4} - 8 + 4 = 0$

Q. 3. Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x, y) on the curve y = f(x) is equal to the cube of the abscissa of P, then find the value of f(–3)

Ans. 9

Solution. The equation of tangent to the curve y = f (x) at the point P (x,y) is

$$\frac{Y-y}{X-x}=\frac{dy}{dx} \text{ or } (X-x)\frac{dy}{dx}-(Y-y)=0$$

$$\Rightarrow X\frac{dy}{dx}-Y=x\frac{dy}{dx}-y$$

$$\text{Its y-intercept} = y-x\frac{dy}{dx}=x^3 \Rightarrow \frac{dy}{dx}-\frac{y}{x}=-x^2$$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx}=\frac{1}{x}$$

$$\therefore y.\frac{1}{x}=\int -x^2\frac{1}{x}dx=\frac{-x^2}{2}+C,\ y=\frac{-x^3}{2}+Cx$$

$$\text{As } f(1)=1 \Rightarrow At\ x=1, y=1$$

$$\therefore 1=\frac{-1}{2}+C \Rightarrow C=3/2 \quad \therefore y=-\frac{x^3}{2}+\frac{3x}{2}$$

$$\text{At } x=-3,\ y=\frac{27}{2}-\frac{9}{2}=9$$

$$\therefore f(-3)=9.$$

Q. 4. Let f be a function defined on R (the set of all real numbers) such that f '(x)=2010 (x–2009) (x–2010)^2 (x–2011)^3 (x–2012)^4 .for all x ∈ R. If g is a function defined on R with values in the interval (0, ∞) such that f (x)=ln (g (x)), for all x ∈ R then the number of points in R at which g has a local maximum is

Ans. 1

Solution. We have,

f '(x) = 2010 (x – 2009)(x – 2010)^2 (x – 2011)^3 (x – 2012)^4

As f (x) = ln g (x) ⇒ g (x) = $e^{f(x)}$ ⇒ g '(x) = $e^{f(x)}$. f '(x)

For max/min, g '(x) = 0 ⇒ f '(x) = 0

Out of two points one should be a point of maxima and other that of minima.

∴ There is only one point of local maxima.

Q. 5. Let f: IR → IR be defined as f (x) = |x| +|x^2- 1|. The total number of points at which f attains either a local maximum or a local minimum is

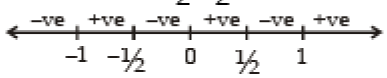
Ans. 5

Solution. We have f (x) = |x| +|x^2- 1|

$$= \begin{cases} -x+x^2-1, & x<-1 \\ -x-x^2+1, & -1\le x\le 0 \\ x-x^2+1, & 0<x<1 \\ x^2+x-1, & x\ge 1 \end{cases}$$

$$\text{We have } f'(x)=\begin{bmatrix} 2x-1 & , & x<-1 \\ -2x-1 & , & -1\le x\le 0 \\ -2x+1 & , & 0<x<1 \\ 2x+1 & , & x>1 \end{bmatrix}$$

Critical pts are $\frac{1}{2}, \frac{-1}{2}, -1, 0$ and 1



We observe at five points f ’(x) changes its sign

∴ There are 5 points of local maximum or local minimum.

Q. 6. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p’(0) is

Ans. 9

Solution. ∴ p(x) has a local maximum at x = 1 and a local minimum at x = 3 and p(x) is a real polynomial of least degree

∴ Let p'(x) = k (x – 1)(x – 3) = k(x^2 – 4x + 3)

$$\Rightarrow p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + C$$

Given $p(1) = 6$ and $p(3) = 2$

$$\Rightarrow \frac{4}{3}k + C = 6 \text{ and } 0 + C = 2 \Rightarrow k = 3$$

$$\therefore p'(x) = 3(x-1)(x-3) \Rightarrow p'(0) = 9$$

Q. 7. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If Δ (h) = area of the triangle PQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$,

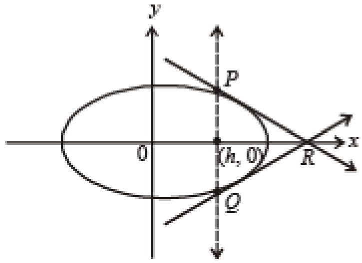
then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

Ans. 9

Solution. Vertical line x = h, meets the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at

$$P\left(h, \frac{\sqrt{3}}{2}\sqrt{4-h^2}\right) \text{ and } Q\left(h, \frac{-\sqrt{3}}{2}\sqrt{4-h^2}\right)$$

By symmetry, tangents at P and Q will meet each other at x-axis.



Tangent at P is $\frac{xh}{4} + \frac{y\sqrt{3}}{6}\sqrt{4-h^2} = 1$

which meets x-axis at $R\left(\frac{4}{h}, 0\right)$

Area of $\Delta PQR = \frac{1}{2} \times \sqrt{3}\sqrt{4-h^2} \times \left(\frac{4}{h} - h\right)$

i.e., $\Delta(h) = \frac{\sqrt{3}}{2}\frac{(4-h^2)^{3/2}}{h}$

$$\frac{d\Delta}{dh} = -\sqrt{3}\left[\frac{\sqrt{4-h^2}(h^2+2)}{h^2}\right] < 0$$

∴ Δ(h) is a decreasing function.

$$\therefore\ \frac{1}{2} \le h \le 1 \Rightarrow \Delta_{max} = \Delta\left(\frac{1}{2}\right) \text{ and } \Delta_{min} = \Delta(1)$$

$$\therefore\ \Delta_1 = \frac{\sqrt{3}}{2}\frac{\left(4-\frac{1}{4}\right)^{3/2}}{\frac{1}{2}} = \frac{45}{8}\sqrt{5}$$

$$\Delta_2 = \frac{\sqrt{3}}{2}\frac{3\sqrt{3}}{1} = \frac{9}{2}$$

$$\therefore\ \frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45 - 36 = 9$$

Q. 8. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is

Ans. 8

Solution.

$$(y-x^5)^2 = x(1+x^2)^2$$

$$2(y-x^5) = \left(\frac{dy}{dx} - 5x^4\right) = (1+x^2)^2 + 2x(1+x^2).2x$$

$$2(3-1)\left(\frac{dy}{dx} - 5\right) = (1+1)^2 + 2(1+1).2 \Rightarrow \frac{dy}{dx} = 8$$

Q. 9. A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container. If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

Ans. 4

Solution. Let r be the internal radius and R be the external radius.

Let h be the internal height of the cylinder.

$\therefore \quad \pi r^2 h = V \Rightarrow h = \dfrac{V}{\pi r^2}$

Also Vol. of material $= M = \pi[(r+2)^2 - r^2]h + \pi(r+2)^2 \times 2$

$$\text{or} \quad M = 4\pi(r+1).\frac{V}{\pi r^2} + 2\pi(r+2)^2$$

$$\Rightarrow \quad M = 4V\left[\frac{1}{r} + \frac{1}{r^2}\right] + 2\pi(r+2)^2$$

$$\frac{dM}{dr} = 4V\left[\frac{-1}{r^2} - \frac{2}{r^3}\right] + 4\pi(r+2)$$

For min. value of M, $\dfrac{dM}{dr} = 0$

$$\Rightarrow \quad \frac{-4V}{r^3}(r+2) + 4\pi(r+2) = 0$$

$$\Rightarrow \quad \frac{4V}{r^3} = 4\pi \text{ or } r^3 = \frac{V}{\pi} = 1000$$

$$\therefore \quad V = 1000\pi$$

$$\therefore \quad \frac{V}{250\pi} = 4$$